

# Introduction to the Standard Model

## William and Mary PHYS 771 Spring 2014

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Class information, including syllabus and homework assignments can be found at  
[http://ntc0.lbl.gov/~walkloud/wm/courses/PHYS\\_771/](http://ntc0.lbl.gov/~walkloud/wm/courses/PHYS_771/)

### Homework Assignment 1

1. We are primarily using the “mostly minus” metric,  $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ . With this metric, the field strength tensor for a classical electromagnetic field is

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix} \quad (1)$$

which can be compactly expressed as  $F_{0i} = E_i$  and  $F_{ij} = -\epsilon_{ijk}B_k$ .

- (a) Beginning with  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ , derive the form of  $F_{\mu\nu}$  if we work with the “mostly plus” metric,  $g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ .
  - (b) Express the space-time  $F_{0i}$  and space-space  $F_{ij}$  components in terms of  $E_i$  and  $B_i$ .
2. We discussed using the covariant derivative to construct the field strength tensors for gauge theories,  $igF_{\mu\nu} = [D_\mu, D_\nu]$ . Suppose we have a fermion that is a doublet that transforms as

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}, \quad \psi(x) \rightarrow e^{i\alpha^a(x)t^a} \psi(x), \quad \text{with } t^a = \frac{\sigma^a}{2}, \quad \sigma_a = \text{Pauli matrices} \quad (2)$$

such that the covariant derivative is

$$D_\mu = \partial_\mu + igA_\mu(x) \quad A_\mu(x) = t^a A_\mu^a(x) \quad (3)$$

- (a) Derive the field strength tensor. You may find it useful to determine the components  $igF_{\mu\nu}^a = [D_\mu, D_\nu]^a$  instead of  $F_{\mu\nu} = F_{\mu\nu}^a t^a$ .
- (b) In terms of the  $A^a$  fields, what is the form of the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\text{tr}[F_{\mu\nu}^2] = -\frac{1}{4}(F_{\mu\nu}^a)^2 = ? \quad (4)$$

3. For a classic electromagnetic field, Eq. (1),

- (a) What is  $F_{\mu\nu}F^{\mu\nu} = ?$
- (b) What is  $\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = ?$  (with the convention  $\epsilon^{0123} = +1$ )

4. For  $U(\lambda) = e^{i\lambda\alpha_a X_a}$  where  $X_a$  are the generators of a Lie Algebra,
- (a) show  $U(\lambda_1)U(\lambda_2) = U(\lambda_1 + \lambda_2)$
5. For  $SU(2)$ , what is the matrix form of the generators
- (a) for the  $j = 1$  representation?
- (b) for the  $j = 3/2$  representation?
6. Dirac algebra. In any representation, the Dirac matrices satisfy the algebra (in 4 dimensions)

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \times \mathbb{1}_{4 \times 4}. \quad (5)$$

In class, we defined the Dirac matrices in the “Dirac Basis”, for which

$$\gamma_D^0 = \begin{pmatrix} \mathbb{1}_{2 \times 2} & 0 \\ 0 & -\mathbb{1}_{2 \times 2} \end{pmatrix}, \quad \gamma_D^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma_D^5 = \begin{pmatrix} 0 & \mathbb{1}_{2 \times 2} \\ \mathbb{1}_{2 \times 2} & 0 \end{pmatrix}, \quad \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3. \quad (6)$$

Another useful and very common basis is the “chiral basis” (or Weyl basis) in which

$$\gamma_\chi^0 = \begin{pmatrix} 0 & \mathbb{1}_{2 \times 2} \\ \mathbb{1}_{2 \times 2} & 0 \end{pmatrix}, \quad \gamma_\chi^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma_\chi^5 = \begin{pmatrix} -\mathbb{1}_{2 \times 2} & 0 \\ 0 & \mathbb{1}_{2 \times 2} \end{pmatrix}, \quad (7)$$

- (a) Determine the similarity transformation which converts from the Dirac to chiral basis

$$\gamma_\chi = S\gamma_D S^{-1} \quad S = ? \quad (8)$$

- (b) What is the similarity transformation that transforms from the chiral to Dirac basis?
- (c) In both the Dirac and chiral basis, in terms of the spinor components, what are

$$\psi_\pm = \frac{1 \pm \gamma^0}{2} \psi = ? \quad (9)$$

- (d) In both the Dirac and chiral basis, in terms of the spinor components, what are

$$\begin{aligned} \psi_R &= \frac{1 + \gamma^5}{2} \psi = ? \\ \psi_L &= \frac{1 - \gamma^5}{2} \psi = ? \end{aligned} \quad (10)$$

7. In class, we discussed the  $g$ -factor for the electron and the nucleons. We saw in general, the elastic electromagnetic structure of a fermion, with parity conserving interactions, can be expressed as

$$\bar{u}(p')\Gamma^\mu(p', p)u(p) = \bar{u}(p') \left[ \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m} F_2(q^2) \right] u(p), \quad q = p' - p, \quad (11)$$

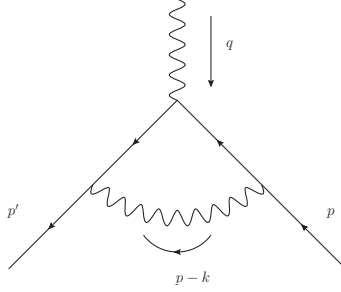


FIG. 1: The Feynman diagram used to compute  $g - 2$  of a point fermion.

where  $u(p)$  is an on-shell fermion spinor which satisfies  $\not{p}u(p) = mu(p)$  and  $\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ . This is the “elastic” structure, because  $\bar{u}(p')$  also represents an on-shell fermion satisfying  $\bar{u}(p)\not{p} = \bar{u}(p)m$ .

In the case of the electron (point-like fermion) we saw the Dirac equation gives  $g = 2 + \frac{\alpha_{f.s.}}{\pi}$ . We noted that for the nucleons,  $g_p \simeq 5.58$  and  $g_n = -3.83$  so that the nucleons are not perturbatively close to point-like fermions, one indication they have interesting internal structure. We commented in class that  $g = 2[F_1(0) + F_2(0)]$  and so for the electron,  $F_2(0) = \frac{\alpha_{f.s.}}{2\pi}$ .

Perform this classic QED calculation, using tools you have been learning in QFT, see Fig. 1. This calculation is so classic, you can easily find the solution in the literature. *I strongly encourage you to attempt it on your own, before resorting external sources, peers, books, etc..* The key to successfully performing this calculation is to realize you isolate the contribution which is proportional to  $\bar{u}(p')\sigma^{\mu\nu}q_\nu u(p)$ . It turns out, this contribution to the diagram in Fig. 1 is free of both Ultraviolet (UV) ( $q_E^2 \rightarrow \infty$ ) and Infrared (IR) ( $q_E^2 \rightarrow 0$ ) singularities (where  $q_E$  is the Euclidean four-momentum obtained after Wick rotation of the momentum integral). To this end, recall the Gordon Identity which can be used to relate  $\bar{u}(p')(p' + p)u(p)$  to  $\bar{u}(p')\sigma^{\mu\nu}q_\nu u(p)$ .

- Compute  $g - 2$  for the electron
- Using just the requirements we have of our QFT, QED (renormalizable, gauge-invariant, Lorentz invariant QFT in 4 space-time dimensions) why should you know ahead of time that the contribution to  $g - 2$  is free of both UV and IR singularities?